Exam # 1 Info and Review

Topics

Material covered in class lectures 8/21 to 9/15. This corresponds to Sections 1.1-1.8, 1.10, 2.1-2.2 in Goodman.

1. Know basic definitions: Group, subgroup, homorphism, isomorphism of groups, subgroup generated by a set, cyclic group, order of an element, Euler's φ-function, gcd, congruence class.
2. Important examples: $\mathbb{Z}, \mathbb{Z}/n\mathbb{Z} = \mathbb{Z}_n, (Z/n\mathbb{Z})^\times = \mathbb{Z}_n^\times, \mathbb{R}, \mathbb{C}, \mathbb{R} \setminus \{0\}, \mathbb{C} \setminus \{0\}, M_n(\mathbb{R}), GL_n(\mathbb{R}), O_n(\mathbb{R}), S_n$.
3. Compute in $\mathbb{Z}_n$: find orders of elements, determine subgroup lattices.
4. Compute in $\mathbb{Z}_n^\times$: determine its elements, find inverses.
5. Compute with permuations: cycle decompostion.
6. Arithmetic of integers: divisibility properties, primes, gcd.
7. Modular arithmetic: addition, multiplications, zero divisors, multiplicative inverses of congruence classes.
8. Polynomial division, irreducibility.

Practice problems

Review homework problems. Here are a few further practice problems.

Question 1 Is $(\mathbb{Z}_5, \cdot)$ a group? What about $(\mathbb{Z}_5^\times, \cdot)$? What about $(\mathbb{Z}_5^\times, +)$?

Question 2 If p is a prime, how many elements does $\mathbb{Z}_p^\times$ have?

Question 3 Suppose that $\varphi(n) = n - 1$. Must n be prime?

Question 4 If $\phi : G \to H$ is an isomorphism of groups, is the inverse function $\phi^{-1} : H \to G$ also an isomorphism of groups?

Question 5 If X is a set and $Y \subseteq X$ is a subset, show that $\{\sigma \in \mathrm{Sym(X)} \mid \sigma|_Y = \mathrm{id}_Y\}$ is a subgroup of $\mathrm{Sym}(X)$ and is isomorphic to $\mathrm{Sym}(X - Y)$.

Question 6 Give an example of a subgroup $H \leq O_2(\mathrm{R})$ such that H is isomorphic to $\mathbb{Z}_4$.

Question 7 Is there an element of order 4 in $\mathbb{Z}_{27}$? Is there an element of order 9?

Question 8 Give an example of two groups of order 6 which are not isomorphic.

Question 9 If $H_1, H_2 \leq G$ are subgroups, show that $H_1 \cap H_2$ is also a subgroup.

Question 10 Is $\mathbb{Z}_5^\times$ cyclic? Is $\mathbb{Z}_9^\times$ cyclic?

Question 11 Write the permutation $(123)(145)$ as a product of disjoint cycles.

Question 12 For each of the following, determine if the polynomial $p(x)$ is irreducible over $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$.

1. $p(x) = x^2 - 2$.
2. $p(x) = x^2 - 4$.
3. $p(x) = x^2 + 1$.